

Conditions for the Steinness of a complex manifold

journal or publication title	大島商船高等専門学校紀要
volume	19
page range	73-79
year	1986-12
URL	http://hdl.handle.net/2298/2968

Conditions for the Steinness of a complex manifold

Makoto Abe*

Abstract

In this paper we give conditions for the Steinness of a complex space or a complex manifold X such that $\dim H^1(X, \mathcal{O}) < +\infty$.

0. Introduction

We may prove the following theorem using the method of J. E. Fornaess – R. Narasimhan [7] and B. Jennane [9], [10].

Let X be a second countable reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.

- i) $H^1(X, \mathcal{O}) = 0$, where \mathcal{O} is the structure sheaf of X .
- ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.*

In case that X is an unramified domain over a Stein manifold, the above statement remains valid even if we replace the condition i) by the weaker one that $\dim H^1(X, \mathcal{O}) < +\infty$. More precisely we have the following result.

Let X be a second countable complex manifold. Then X is a Stein manifold if the following three conditions are satisfied.

- i) $\dim H^1(X, \mathcal{O}) < +\infty$, where \mathcal{O} is the structure sheaf of X .
- ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.*
- iii) *There exists a realization $\Phi : X \rightarrow S$ as a ramified domain over a Stein manifold S such that the ramification locus of it is an empty set or a Stein variety of X .*

In addition to the above two theorems we also give an improvement of a result in the author and M. Furushima [2].

1. Unramified domains over a Stein manifold

Throughout this paper all complex spaces are supposed to be second countable. We always denote by \mathcal{O} the structure sheaf of the complex space X considered in each cases. A reduced closed complex subspace of a complex space X is called a variety of X .

Let X and Z be complex spaces. Let $\Phi : X \rightarrow Z$ be a holomorphic map such that all fibers of Φ are discrete sets in X . Then $\Phi : X \rightarrow Z$ is called a ramified domain over Z . The unique minimal variety V of X such that the map $\Phi|_{(X-V)}$ is locally biholomorphic is called the ramification locus of $\Phi : X \rightarrow Z$. If $V = \emptyset$, then $\Phi : X \rightarrow Z$ is called an unramified domain over Z .

LEMMA 1. Let X be a reduced complex space such that $\dim H^1(X, \mathcal{O}) < +\infty$. Let f be a holomorphic function on X such that the set $Y = \{x \in X \mid f(x) = 0\}$ is a Stein variety of X . Then for every holomorphic function g on Y there exist an integer N and a holomorphic function G on X such that $1 \leq N \leq \dim H^1(X, \mathcal{O}) + 1$ and $G|_Y = g^N$.

PROOF. Y has a Stein neighbourhood U_0 by the main theorem of Y. - T. Siu [11]. We may assume that $g \in \mathcal{O}(U_0)$. $\mathcal{Z} = \{U_0, U_1\}$ is an open covering of X , where $U_1 = X - Y$. $g/f \in \mathcal{O}(U_0 \cap U_1) = \mathcal{Z}^1(\mathcal{Z}, \mathcal{O})$. Since the natural map $H^1(\mathcal{Z}, \mathcal{O}) \rightarrow H^1(X, \mathcal{O})$ is injective, $\dim H^1(\mathcal{Z}, \mathcal{O}) \leq \dim H^1(X, \mathcal{O}) < +\infty$. Let N be a minimal integer such that $\{[(g/f)^k]\}_{k=1}^N$ is linearly dependent in $H^1(U, \mathcal{O})$. Then $1 \leq N \leq \dim H^1(X, \mathcal{O}) + 1$. There exist $c_1, \dots, c_{N-1} \in \mathbb{C}$, $h_0 \in \mathcal{O}(U_0)$ and $h_1 \in \mathcal{O}(U_1)$ such that $\sum_{k=1}^{N-1} c_k (g/f)^k + (g/f)^N = h_1 - h_0$ on $U_0 \cap U_1 = U_0 - Y$. We can define $G \in \mathcal{O}(X)$ by the equations $G = \sum_{k=1}^{N-1} c_k g^k f^{N-k} + g^N + h_0 f^N$ on U_0 and $G = h_1 f^N$ on U_1 . Then $G = g^N$ on Y .

LEMMA 2. Let $\Phi : X \rightarrow S$ be an unramified domain over a Stein manifold S . Then X is a Stein manifold if the following two conditions are satisfied.

- i) $\dim H^1(X, \mathcal{O}) < +\infty$.
- ii) There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.

PROOF. Without loss of generality we may assume that S is connected and of dimension n . In the proof we use the method of the proof of Theorem 7 of the author and Y. Abe [1]. Suppose that $\Phi : X \rightarrow S$ is not p_7 -convex in the sense of F. Docquier and H. Grauert [6]. Then there exist H, P, φ and ψ with following properties. $H = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| \leq 1 (j = 1, \dots, n)\} \cup \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid 1 - \varepsilon < |z_1| < 1 + \varepsilon, |z_j| < 1 + \varepsilon (j = 2, \dots, n)\}$, $P = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid |z_j| < 1 + \varepsilon (j = 1, \dots, n)\}$, $0 < \varepsilon < 1$. $\varphi : \bar{H} \rightarrow \tilde{X} = X \cup \tilde{\partial} X$ is a continuous map such that $\varphi(H) \subset X$. There exists $b = (b_1, \dots, b_n)$ such that $|b_1| < 1 - \varepsilon$, $\max_{2 \leq j \leq n}$

$|b_j| = 1$ and $\varphi(b) \in \widetilde{\partial}X$. Here $\widetilde{\partial}X$ denotes the ideal boundary of $\Phi : X \rightarrow S$ defined in F. Docquier and H. Grauert [6]. $\psi : P \rightarrow S$ is an injective holomorphic map such that $\Phi \circ (\varphi|_H) = \psi|_H$. Let $\lambda : X \rightarrow X'$, $\Phi' : X' \rightarrow S$ be an envelope of holomorphy of $\Phi : X \rightarrow S$. It holds that $\Phi' \circ (\lambda \circ (\varphi|_H)) = \psi|_H$. Since X' is a Stein manifold, there exists a holomorphic map $\varphi' : P \rightarrow X'$ such that $\varphi'|_H = \lambda \circ (\varphi|_H)$. $\varphi'(b)$ is a boundary point of $\lambda(X)$ in X' . There exists a holomorphic function f' on X' such that $f' \circ \lambda = f$. Put $t_0 = f'(\varphi'(b))$. Take an irreducible component A containing b of the variety $\{z \in P \mid (f' \circ \varphi')(z) = t_0\}$ of P . By the pseudoconvexity of $P - A$ it holds that $H \cap A \neq \emptyset$. Let $\gamma : [0, 1] \rightarrow A$ such that $\gamma(0) \in H \cap A$ and $\gamma(1) = b$. Since $\varphi'(\gamma(0)) \in \lambda(X)$ and $\varphi'(\gamma(1)) = \varphi'(b) \notin \lambda(X)$, there exist a sequence $(q_\nu)_{\nu=1}^\infty \subset X$ and a boundary point p_0 of $\lambda(X)$ in X' such that $\lim_{\nu \rightarrow \infty} \lambda(q_\nu) = p_0$ and $\lambda(q_\nu) \in \varphi'(\gamma([0, 1]))$. Since $Y = \{x \in X \mid f(x) = t_0\}$ is Stein and $(q_\nu)_{\nu=1}^\infty$ is a divergent sequence in Y , there exists a holomorphic function g on Y which satisfies $\lim_{\nu \rightarrow \infty} |g(q_\nu)| = +\infty$. By Lemma 1 there exist an integer $N \geq 1$ and a holomorphic function G on X such that $G|_Y = g^N$. There exists a holomorphic function G' on X' such that $G' \circ \lambda = G$. It holds that $|G'(p_0)| = \lim_{\nu \rightarrow \infty} |g(q_\nu)|^N = +\infty$. It is a contradiction. Thus X is a Stein manifold by Satz 10 of F. Docquier and H. Grauert [6].

COROLLARY. *Let X be an open set of \mathbb{C}^n such that $\dim H^1(X, \mathcal{O}) < +\infty$. Suppose that $X \cap \{z_1 = t\}$ is Stein for every $t \in \mathbb{C}$. Then X is a Stein manifold.*

2. Complex spaces with a vanishing cohomology group

LEMMA 3. *Let X be a reduced complex space. Then X is holomorphically separable if the following two conditions are satisfied.*

- i) $\dim H^1(X, \mathcal{O}) < +\infty$.
- ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.*

PROOF. Let x_0 and x_1 be two distinct points in X . In case that $f(x_0) = f(x_1)$, we put $t_0 = f(x_0)$. $Y = \{x \in X \mid f(x) = t\}$ is Stein and $x_0, x_1 \in Y$. There exists a holomorphic function g on Y such that $g(x_0) = 0$ and $g(x_1) = 1$. By Lemma 1 there exist an integer $N \geq 1$ and a holomorphic function G on X such that $G|_Y = g^N$. $G(x_0) = g(x_0)^N = 0$, $G(x_1) = g(x_1)^N = 1$. It follows that X is holomorphically separable.

To prove Theorem 1 we use the following Lemmas 4 and 5 which is originally due to J. E. Forneaess and R. Narasimhan [7].

LEMMA 4. Let X be a reduced complex space of finite dimension such that $H^1(X, \mathcal{O}) = 0$. Let f be a holomorphic function on X such that the set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and an open set $U \subset X$ such that $U \cap Y_t$ is a Runge Stein open set of Y_t and $K \cap Y_t \subset U \cap Y_t$ for $t \in \mathbb{C}$, $|t| < \varepsilon$.

PROOF. See the proof of Lemma 2.4 of J. E. Fornæss and R. Narasimhan [7]. Instead of Lemma 2.2 of [7] we use Lemma 1.

LEMMA 5. Let X be a reduced complex space of finite dimension without isolated points such that $H^1(X, \mathcal{O}) = 0$. Let f be a holomorphic function on X which is not constant on any irreducible component of X and assume that the set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$. Then for every compact set $K \subset X$ there exist $\varepsilon > 0$ and a compact set $L \subset X$, $K \subset L$, such that the following conditions are satisfied: for $t \in \mathbb{C}$, $|t| < \varepsilon$, there exists $M_t > 0$ such that for any $\tau > 0$ and for any holomorphic function g on Y_t there exists a holomorphic function G on X which satisfies $G|_{Y_t} = g$ and $\|G\|_K \leq M_t \|g\|_{L \cap Y_t} + \tau$.

PROOF. We can prove this lemma along the lines of the proof of Lemme 4 in B. Jennane [9].

THEOREM 1. Let X be a reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.

- i) $H^1(X, \mathcal{O}) = 0$.
- ii) There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.

PROOF. In the proof we use the method of B. Jennane [10]. We first deal with the case that f is not constant on any irreducible component of X . Since X is holomorphically separable by Lemma 3, it suffices to prove that X is holomorphically convex. Let K be an arbitrary compact set in X and $s \in \mathbb{C}$. Put $Y_t = \{x \in X \mid f(x) = t\}$ for $t \in \mathbb{C}$. Then there exist $\varepsilon > 0$ and a compact set $L \subset X$ such that the statement of Lemma 5 holds for $f - s$ instead of f . Take $p \in Y_t - (L \cap Y_t)^\wedge$ for $t \in \mathbb{C}$, $|t - s| < \varepsilon$. There exists a holomorphic function g on Y_t such that $|g(p)| > 1$ and $\|g\|_{L \cap Y_t} < 1/(2M_t)$. There exists $G \in \mathcal{O}(X)$ such that $G|_{Y_t} = g$, $\|G\|_K \leq M_t \|g\|_{L \cap Y_t} + 1/2 < 1$. Since $|G(p)| = |g(p)| > 1$, $p \notin \hat{K}_X$. Therefore $\hat{K}_X \cap Y_t \subset (L \cap Y_t)^\wedge$ for $t \in \mathbb{C}$, $|t - s| < \varepsilon$. By Lemma 4 there exists $\delta(s) > 0$ such that $\delta(s) < \varepsilon$ and $\bigcup_{|t - s| < \delta(s)} Y_t$

$(L \cap Y_t)^\wedge \subseteq X$. Then we have $\hat{K}_X \cap \{x \in X \mid |f(x) - s| < \delta(s)\} = \bigcup_{|t-s| < \delta(s)} (t-s)$
 $(\hat{K}_X \cap Y_t) \subseteq X$. There exists finitely many $s_1, \dots, s_N \in \mathbb{C}$ such that $\{t \in \mathbb{C} \mid |t| \leq \|f\|_K\} \subseteq \bigcup_{j=1}^N \{t \in \mathbb{C} \mid |t - s_j| < \delta(s_j)\}$. Since $\hat{K}_X \subseteq \{x \in X \mid |f(x)| \leq \|f\|_K\}$, $\hat{K}_X = \bigcup_{j=1}^N (\hat{K}_X \cap \{x \in X \mid |f(x) - s_j| < \delta(s_j)\}) \subseteq X$. It follows that X is holomorphically convex.

We now deal with the general case. Let X' be the union of irreducible components of X on which f is constant and X'' be the union of those on which f is not constant. Since every irreducible component of X' is a variety of the Stein space $\{x \in X \mid f(x) = c\}$ for some $c \in \mathbb{C}$, X' is a Stein variety of X . Let \mathcal{J}' and \mathcal{J}'' be the defining ideals of the varieties X' and X'' respectively. We have the exact sequence $0 \rightarrow \mathcal{O} \xrightarrow{\rho} (\mathcal{O}/\mathcal{J}') \oplus (\mathcal{O}/\mathcal{J}'') \xrightarrow{\tau} \mathcal{O}/(\mathcal{J}' + \mathcal{J}'')$, where ρ is the homomorphism $h_x \rightarrow (h_x + \mathcal{J}'_x, h_x + \mathcal{J}''_x)$ and τ is the homomorphism $(g_x + \mathcal{J}'_x, h_x + \mathcal{J}''_x) \rightarrow g_x - h_x + (\mathcal{J}'_x + \mathcal{J}''_x)$. From this we deduce the exact sequence of cohomology groups $H^1(X, \mathcal{O}) \rightarrow H^1(X', \mathcal{O}_{X'}) \oplus H^1(X'', \mathcal{O}_{X''}) \rightarrow H^1(X, \mathcal{O}/(\mathcal{J}' + \mathcal{J}''))$, where $\mathcal{O}_{X'} = (\mathcal{O}/\mathcal{J}')|_{X'}$ and $\mathcal{O}_{X''} = (\mathcal{O}/\mathcal{J}'')|_{X''}$. $\text{supp}(\mathcal{O}/(\mathcal{J}' + \mathcal{J}'')) = X' \cap X'' \subseteq X'$. Since X' is Stein, $H^1(X', \mathcal{O}_{X'}) = 0$ and $H^1(X, \mathcal{O}/(\mathcal{J}' + \mathcal{J}'')) = H^1(X', (\mathcal{O}/(\mathcal{J}' + \mathcal{J}''))|_{X'}) = 0$. $H^1(X, \mathcal{O}) = 0$. Therefore $H^1(X'', \mathcal{O}_{X''}) = 0$. Then by what was shown above X'' is Stein. It follows $X = X' \cup X''$ is Stein.

Using Theorem 1 we obtain the following result which is an improvement of Theorem 8 of the author and M. Furushima [2].

THEOREM 2. *Let X be a K -complete reduced complex space of finite dimension. Then X is a Stein space if the following two conditions are satisfied.*

- i) $\dim H^1(X, \mathcal{O}) < +\infty$.
- ii) *For every holomorphic function f on X which is not constant on any positive dimensional irreducible component of X , the set $\{x \in X \mid f(x) = 0\}$ is a Stein variety of X .*

PROOF. Without loss of generality we may assume that X has no isolated point. Let F be the set of all holomorphic functions on X which are not constant on any irreducible component of X . Since X is K -complete, $F \neq \emptyset$. Every $f \in F$ satisfies the condition ii) of Theorem 1, so it suffices to prove that $H^1(X, \mathcal{O}) = 0$ using the method of E. Ballico [5]. Let $f \in F$. Since the set $Y = \{x \in X \mid f(x) = 0\}$ is nowhere dense in X , the multiplication by f defines the monomorphism $m(f): \mathcal{O} \rightarrow \mathcal{O}$. The exact sequence $0 \rightarrow \mathcal{O} \xrightarrow{m(f)} \mathcal{O} \rightarrow \mathcal{O}/f\mathcal{O} \rightarrow 0$ induces the exact sequence of cohomology groups $H^1(X, \mathcal{O})$

$m(f)^* : H^1(X, \mathcal{O}) \rightarrow H^1(X, \mathcal{O}/f\mathcal{O})$. Since the complex space $(Y, (\mathcal{O}/f\mathcal{O})|_Y)$ is Stein by a theorem of H. Grauert (Theorem 5 of [8], p. 154), $H^1(X, \mathcal{O}/f\mathcal{O}) = H^1(Y, (\mathcal{O}/f\mathcal{O})|_Y) = 0$. Therefore $m(f)^*$ is surjective. Since $H^1(X, \mathcal{O})$ is of finite dimension, $m(f)^*$ is an isomorphism. Suppose that $H^1(X, \mathcal{O}) \neq 0$ and take $\xi \in H^1(X, \mathcal{O})$, $\xi \neq 0$. There exists $(a_1, \dots, a_N) \in \mathbb{C}^N - \{0\}$ such that $\sum_{j=1}^N a_j m(f^j)^*(\xi) = 0$, where $N = \dim H^1(X, \mathcal{O}) + 1$. Since $\sum_{j=1}^N a_j f^j \in F$, $\sum_{j=1}^N a_j m(f^j)^*(\xi) = m(\sum_{j=1}^N a_j f^j)^*(\xi) \neq 0$. It is a contradiction. It follows that $H^1(X, \mathcal{O}) = 0$.

Theorem 1 was obtained implicitly by B. Jennane [10] in the proof of his main result. Using the method of E. Ballico [5] we reduced Theorem 2 to Theorem 1. But in the category of K -complete reduced complex spaces the condition ii) in Theorem 2 is stronger than the condition ii) in Theorem 1.

3. Remified domains over a Stein manifold

We give a condition for the Steinness of a complex manifold which includes Lemma 2 as a special case.

THEOREM 3. *Let X be a complex manifold. Then X is a Stein manifold if the following three conditions are satisfied.*

- i) $\dim H^1(X, \mathcal{O}) < +\infty$.
- ii) *There exists a holomorphic function f on X such that the set $\{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$.*
- iii) *There exists a realization $\Phi: X \rightarrow S$ as a ramified domain over a Stein manifold S such that the ramification locus of it is an empty set or a Stein variety of X .*

PROOF. Without loss of generality we may assume that X is connected. Let V be the ramification locus of $\Phi: X \rightarrow S$. V is a variety of X of pure codimension 1. Since V is Stein, there exists a Stein open set U_0 which includes V by the main theorem of Y.-T. Siu [11]. $\mathcal{Z} = \{U_0, U_1\}$ is an open covering of X , where $U_1 = X - V$. By Satz 1 of F. Docquier and H. Grauert [6], $U_0 \cap U_1 = U_0 - V$ is a Stein manifold. From the Mayer-Vietoris exact sequence $\cdots \rightarrow H^1(X, \mathcal{O}) \rightarrow H^1(U_0, \mathcal{O}) \oplus H^1(U_1, \mathcal{O}) \rightarrow H^1(U_0 \cap U_1, \mathcal{O}) \rightarrow \cdots$ we have a surjection $H^1(X, \mathcal{O}) \rightarrow H^1(U_1, \mathcal{O})$. $\dim H^1(U_1, \mathcal{O}) \leq \dim H^1(X, \mathcal{O}) < +\infty$. $\Phi|_{U_1}: U_1 \rightarrow S$ is an unramified domain over S . The set $Y_t = \{x \in X \mid f(x) = t\}$ is a Stein variety of X for every $t \in \mathbb{C}$. $Y_t \cap U_1 = \{x \in U_1 \mid f(x) = t\} = Y_t - V$. $Y_t \cap U_0$ is a Stein neighbourhood of the boundary of an open set $Y_t \cap U_1$ in the

reduced Stein space Y_t . $(Y_t \cap U_0) \cap (Y_t \cap U_1) = Y_t \cap (U_0 - V)$ is Stein. Therefore $Y_t \cap U_1$ is a Stein open set of Y_t by Theorem 4 of A. Andreotti and R. Narasimhan [4]. It follows that U_1 is Stein by Lemma 2. From the Mayer-Vietoris exact sequence $\cdots \rightarrow H^{i-1}(U_0 \cap U_1, \mathcal{O}) \rightarrow H^i(X, \mathcal{O}) \rightarrow H^i(U_0, \mathcal{O}) \oplus H^i(U_1, \mathcal{O}) \rightarrow \cdots$ we have that $H^i(X, \mathcal{O}) = 0$ for $i \geq 2$. Then $H^1(X, \mathcal{O}) = 0$ by the theorem of L. Alessandrini [3] and Lemma 3. It follows that X is a Stein manifold by Theorem 1.

References

- [1] M. ABE and Y. ABE, *Domains over a K -complete manifold*, Mem. Fac. Sci. Kyushu Univ. **38** (1984), 133–140.
- [2] M. ABE and M. FURUSHIMA, *Note on the theorem of Cartan–Behnke–Stein*, Mem. Fac. Sci. Kyushu Univ. **37** (1983), 57–62.
- [3] L. ALESSANDRINI, *On the cohomology of a holomorphically separable complex analytic space*, Boll. Un. Mat. Ital. **6** (1982), 261–268.
- [4] A. ANDREOTTI and R. NARASIMHAN, *Oka's Heftungslemma and the Levi problem for complex spaces*, Trans. Amer. Math. Soc. **111** (1964), 345–366.
- [5] E. BALLICO, *A remark on a theorem by Forneaess and Narasimhan*, Math. Ann. **253** (1980), 137–138.
- [6] F. DOCQUIER and H. GRAUERT, *Levisches Problem und Rungescher Satz für Teilgebiete Steinscher Mannigfaltigkeiten*, Math. Ann. **140** (1960), 94–123.
- [7] J. E. FORNAESS and R. NARASIMHAN, *The Levi problem on complex spaces with singularities*, Math. Ann. **248** (1980), 47–72.
- [8] H. GRAUERT and R. REMMERT, *Theory of Stein spaces*, Grundle. 236, Springer–Verlag, Berlin Heidelberg New York, 1979.
- [9] B. JENNANE, *Problème de Levi et morphisme localement de Stein*, Math. Ann. **256** (1981), 37–42.
- [10] B. JENNANE, *Problème de Levi et espaces holomorphiquement séparés*, Math. Ann. **268** (1984), 305–316.
- [11] Y.-T. SIU, *Every Stein subvariety admits a Stein neighborhood*, Invent. math. **38** (1976), 89–100.